Journal of Scientific Research
Banaras Hindu University, Varanasi
Vol. 57, 2013 : 165-167
ISSN : 0447-9483

Lanczos derivative applied to Fourier series

R. Cruz-Santiago, J. López-Bonilla, R. López-Vázquez,
ESIME-Zacatenco, ICE, Instituto Politécnico Nacional,
Edif.5, 1er. Piso, Col. Lindavista, CP 07738, México DF,
jlopezb@ipn.mx

Abstract.

It is very known that if the operator $\frac{d}{dx}$ acts on each term into a convergent Fourier series (FS), then it may result a divergent series. This situation is remedied applying the symmetric derivative to FS, which implies the existence of the important Fejér-Lanczos Factors. In this note, we show that the Lanczos derivative also leads to these Factors.

1.- Introduction.

If on the Fourier series [1]:

$$f(x) = \frac{1}{2}\, a_0 + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \;, \tag{1}$$

convergent in $[-\pi, \pi]$, we apply the operator $\frac{d}{dx}$ results: $\frac{d}{dx} f(x) = \sum_{k=1}^{\infty} k\,[-a_k \sin(kx) + b_k \cos(kx)]$, (2) which it may be divergent [2,3]. This problem was remedied by Lanczos [3] with $f'(x)$ defined as a Symmetric Derivative:

$$f'(x) \equiv \lim_{n\to\infty} \frac{1}{\frac{2\pi}{n}} \left[f_n\left(x + \frac{\pi}{n}\right) - f_n\left(x - \frac{\pi}{n}\right) \right] , \tag{3}$$

with the partial sums:

$$f_n(x) = g_n(x) + h_n(x) \;,$$

$$g_n(x) = \frac{1}{2}\, a_0 + \sum_{k=1}^{n} a_k \cos(kx) \;,\; h_n(x) = \sum_{k=1}^{n} b_k \sin(kx) \;, \tag{4}$$

resulting the convergent expression:

$$f'(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \sigma_k \frac{d}{dx} [a_k \cos(kx) + b_k \sin(kx)] , \tag{5}$$

with the Fejér-Lanczos Factors [3,4]:

$$\sigma_0 = 1 \;,\; \sigma_k = \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}} \;,\; k = 1, \ldots, n \;,\; \sigma_n = 0 \;. \tag{6}$$

The set of factors σ_k , for a given n, it is equivalent to a discrete sampling function.

In (2) and (3), we employ two derivatives, however, also there is the Lanczos derivative [3, 5-11], then it is natural to ask if this ultimate derivative leads to relation (5). The answer is yes, to see the next section.

2.- Lanczos generalized derivative.

Lanczos [3] used the least square method of Gauss-Legendre to obtain an integral expression for the derivative of a function, that is, differentiation by integration:

$$F'(x) = \lim_{\epsilon \to 0} \frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, F(x+t)\, dt , \tag{7}$$

which it may be applied to Fourier case:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, g_n(x+t)\, dt \overset{(4)}{=} \frac{3}{2\epsilon^3} \sum_{k=1}^{n} a_k \int_{-\epsilon}^{\epsilon} t\, \cos(kx+kt)\, dt ,$$

$$= -3 \sum_{k=1}^{n} a_k \frac{\sin(kx)}{k^2} A_k \quad \text{with} \quad A_k(\epsilon) = \frac{1}{\epsilon^3} [\sin(k\epsilon) - k\,\epsilon \cos(k\epsilon)] ; \tag{8}$$

similarly:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, h_n(x+t)\, dt \overset{(4)}{=} \frac{3}{2\epsilon^3} \sum_{k=1}^{n} b_k \int_{-\epsilon}^{\epsilon} t\, \sin(kx+kt)\, dt ,$$

$$= 3 \sum_{k=1}^{n} b_k \frac{\cos(kx)}{k^2} A_k . \tag{9}$$

Therefore, the Lanczos derivative applied to partial sum (4) gives, taking $\epsilon = \frac{\pi}{n}$:

$$f'(x) = \lim_{n \to \infty} \frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t\, f_n(x+t)\, dt ,$$

$$\overset{(8)\,and\,(9)}{=} \lim_{n \to \infty} 3 \sum_{k=1}^{n} \frac{1}{k^2} A_k [-a_k \sin(kx) + b_k \cos(kx)] ,$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{3A_k}{k^3} \frac{d}{dx} [a_k \cos(kx) + b_k \sin(kx)] , \tag{10}$$

but the Bernoulli–Hôpital rule permits to observe the behavior:

$$A_k\left(\epsilon = \frac{\pi}{n}\right) \xrightarrow[n>>1]{} \frac{k^3}{3} \frac{\sin(k\,\epsilon)}{k\,\epsilon} = \frac{k^3}{3} \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}} \overset{(6)}{=} \frac{k^3}{3} \sigma_k , \tag{11}$$

and this value employed in (10) implies (5), q.e.d.

Thus, it is proved that the Symmetric and Lanczos derivatives give us the same expression for the derivative of a infinite Fourier series, with the important participation of the Fejér–Lanczos factors.

References

C. Lanczos, Discourse on Fourier series, Oliver & Boyd, London (1966)

J. W. Gibbs, Fourier´s series, Nature 59 (1898) 200

C. Lanczos, Applied analysis, Prentice-Hall, New Jersey (1956)

C. Lanczos, Linear differential operators, D. Van Nostrand Co., London (1961)

C. W. Groetsch, Lanczos generalized derivative, Am. Math. Monthly 105, No.4 (1998) 320-326

J. Shen, On the Lanczos generalized derivative, Am. Math. Monthly 106, No.8 (1999) 766-768

D. L. Hicks, L.M. Liebrok, Lanczos generalized derivative: Insights and applications, Applied Maths. and Compt. 112, No.1 (2000) 63-73

N. Burch, P. E. Fishback, R. Gordon, The least-squares property of the Lanczos derivative, Maths. Mag. 78, No.5 (2005) 368-378

Lizzy Washburn, The Lanczos derivative, Senior Project Archive, Dept. of Maths., Whitman College, USA (2006)

J. López-Bonilla, J. Rivera R., S.Vidal B., Lanczos derivative via a quadrature method, Int. J. Pure Appl. Sci. Technol. 1, No.2 (2010) 100-103

J. López–Bonilla, A. Rangel M., A. Zuñiga–Segundo, Derivada generalizada de Lanczos en una discontinuidad finita, Ini. Inv. (Univ. of Jaén, Spain) No.5: a4 (2010) 1-5